GOVERNMENT OF ANDHRA PRADESH

COMMISSIONERATE OF COLLEGIATE EDUCATION

INVERSE LAPLACE TRANSFORMS - II

DIVISION BY POWERS OF ‘P’

MATHEMATICS

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Inverse Laplace Transformation – Division by powers of p

Learning Objectives

Students will be able to recognize the following properties of the Inverse Laplace Transformation.

- Understand Inverse Laplace transforms of standard functions.
- Get an idea about division by powers of p.
- Get the knowledge of application of Laplace transformations.
- Students will be able to know the use of Inverse Laplace transform in system modeling, digital signal processing, etc.,

Definition of Inverse Laplace Transformation

If $f(p)$ is the Laplace Transform of a function $F(t)$. i.e. $L\{ F(t) \} = f(p)$ then $F(t)$ is called the inverse Laplace transform of the function $f(p)$ and is written as $F(t) = L^{-1} \{ f(p) \}$.

Inverse Laplace Transform of Division by Powers of p

Theorem: $\boldsymbol{If\, L^{-1}\{f(p)] = F(t)\ then\ L^{-1}\left\{\frac{f(p)}{p}\right\} = \int_0^t F(t)dt}$

Proof: $Let\ G(t) = \int_0^t F(t)dt.\ Then\ G'(t) = F(t) and\ G(0) = 0$

$\therefore L(G'(t)) = p\, L\big(G(t)\big) - G(0)$

$$f(p) = p\, L\big(G(t)\big)$$

$i.e\ L\big(G(t)\big) = \frac{f(p)}{p}$

$i.e\ L^{-1}\left(\frac{f(p)}{p}\right) = G(t) = \int_0^t F(t)dt.$

Theorem: $\boldsymbol{If\, L^{-1}\{f(p)] = F(t) \; then \; L^{-1}\left\{\frac{f(p)}{p^2}\right\} = \int_0^t \int_0^y F(x)dx\, dy}$

Proof: Let$\mathrm{G(t)} = \int_0^t \int_0^y f(x)dx\, dy$

$Then\; G'(t) = \int_0^t F(x)dx, \;\; where\; G''(t) = F(t).$

$Also\; G(0) = G'(0) = 0$

Now $\mathrm{L}(G''(t)) = p^2L\{G(t)\} - pG(0) - G'(0) = p^2L\{G(t)\}$

$\Rightarrow f(p) = p^2L\{G(t)\}$

$\Rightarrow L\{G(t)\} = \frac{f(p)}{p^2}$

$\Rightarrow L^{-1}\left\{\frac{f(p)}{p}\right\} = G(t) = \int_0^t \int_0^y f(x)dx\, dy$

Ex 1: Find the inverse Laplace transform of $\frac{1}{p(p+2)}$

Sol: $L^{-1}\left\{\frac{1}{p+2}\right\} = e^{-2t} = F(t)$

we have $L^{-1}\left\{\frac{1}{p(p+2)}\right\} = L^{-1}\left\{\frac{1}{p}\left(\frac{1}{p+2}\right)\right\}$

$$= \int_0^t e^{-2t}\,dt$$

$$= \left(\frac{e^{-2t}}{-2}\right)_0^t$$

$$= \frac{1}{2}\left(1 - e^{-2t}\right)$$

$$L^{-1}\left\{\frac{1}{p(p+2)}\right\} = \frac{1}{2}\left(1 - e^{-2t}\right)$$

Ex 2: **Find the inverse Laplace transform of** $\left\{\frac{1}{p(p^2+a^2)}\right\}$

Sol: We have $L^{-1}\left\{\frac{1}{(p^2+a^2)}\right\} = \frac{1}{a}\sin at$ and

$$L^{-1}\left\{\frac{1}{p}F(p)\right\} = \int_0^t F(u)du \ \ where\ F(t) = L^{-1}\{F(p)\}$$

$$\therefore L^{-1}\left\{\frac{1}{p(p^2+a^2)}\right\} = \int_0^t \frac{1}{a}\sin au \ \ du = -\frac{1}{a}\ \ (1-\cos au)_0^t = \frac{1-\cos at}{a^2}$$

Ex 3: **Find the inverse Laplace transform of** $\frac{1}{p^3(p^2+a^2)}$

Sol: We have $L^{-1}\left\{\frac{1}{p(p^2+a^2)}\right\} = \frac{1}{a^2}(1-\cos at)$

$$\therefore L^{-1}\left\{\frac{1}{p^2(p^2+a^2)}\right\} = \int_0^t \frac{1}{a^2}(1-\cos at)dt = \frac{1}{a^2}\left(t-\frac{1}{a}\sin at\right)$$

$$\therefore L^{-1}\left\{\frac{1}{p^3(p^2+a^2)}\right\} = \int_0^t \frac{1}{a^2}\left(t-\frac{1}{a}\sin at\right)dt = \frac{1}{a^2}\left(\frac{t^2}{2}+\frac{1}{a^2}\ (\cos at-1)\right).$$

Ex 4: Find the inverse Laplace transform of $\frac{1}{p^3(p^2+1)}$

Sol: Put a = 1 in the above problem we get

$$L^{-1}\left\{\frac{1}{p^3(p^2+1)}\right\} = \frac{t^2}{2} + \cos t \; - 1$$

Ex 5: Find $L^{-1}\left\{\frac{1}{\left(p^2+a^2\right)^2}\right\}$

Sol: Let $f(p) = \frac{\mathrm{p}}{(\mathrm{p}^2+\mathrm{a}^2)^2}$ $.\,we\ know\ that\ L\{\sin at\} = \frac{a}{p^a+a^2}$

$$Now\ L^{-1}\left\{\frac{d}{dp}\left(\frac{1}{p^2+a^2}\right)\right\} = (-1)t\frac{1}{a}\sin at$$

$$\Rightarrow L^{-1}\left\{\left(\frac{-2p}{p^2+a^2}\right)\right\} = \frac{-t}{a}\sin at$$

$$\Rightarrow L^{-1}\left\{\left(\frac{p}{p^2+a^2}\right)\right\} = \frac{t}{2a}\sin at$$

$$Now\ L^{-1}\left\{\frac{1}{(p^2+a^2)^2}\right\} = L^{-1}\left\{\frac{1}{p}\frac{p}{(p^2+a^2)^2}\right\} = L^{-1}\left\{\frac{1}{p}f(p)\right\}$$

$$= \int_0^t \frac{f(u)}{du} = \int_0^t \frac{u}{2a}\sin au\, du$$

$$= \frac{1}{2a}\int_0^t u\sin au\, du$$

$$= \frac{1}{2a}\left[u\left(\frac{-\cos au}{u}\right) - 1\left(\frac{-\sin au}{a^2}\right)\right]_0^t$$

$$= \frac{1}{2a}\left[\left(\frac{-u\cos au}{u}\right) + \left(\frac{\sin au}{a^2}\right)\right]_0^t$$

$$= \frac{1}{2a^3}[\sin au - au\cos au]_0^t$$

$$= \frac{1}{2a^3}(\sin at - at\cos at)$$

$$\therefore L^{-1}\left\{\frac{1}{(p^2+a^2)^2}\right\} = \frac{1}{2a^3}(\sin at - at\cos at)$$

Ex 6: Find $L^{-1}\left\{\frac{p^2}{(p^2+a^2)^2}\right\}$

Sol: $\mathrm{L}^{-1}\left\{\frac{\mathrm{p}^2}{(\mathrm{p}^2+\mathrm{a}^2)^2}\right\} = \mathrm{L}^{-1}\left\{\mathrm{p}\,\frac{\mathrm{p}}{(\mathrm{p}^2+\mathrm{a}^2)^2}\right\} = \frac{d}{dt}\mathrm{L}^{-1}\left\{\frac{\mathrm{p}}{(\mathrm{p}^2+\mathrm{a}^2)^2}\right\} \ldots\ldots\ldots.(1)$

We know that $\mathrm{L}^{-1}\left\{\frac{1}{\mathrm{p}^2+\mathrm{a}^2}\right\} = \frac{1}{a}\sin at \ldots\ldots\ldots(2)$

We have $\frac{d}{dp}\left\{\frac{1}{\mathrm{p}^2+\mathrm{a}^2}\right\} = \frac{-2p}{(\mathrm{p}^2+\mathrm{a}^2)^2}$

Then $L^{-1}\left[\frac{d}{dp}\left\{\frac{1}{\mathrm{p}^2+\mathrm{a}^2}\right\}\right] = (-1)tL^{-1}\left\{\frac{1}{\mathrm{p}^2+\mathrm{a}^2}\right\}$

$\Rightarrow L^{-1}\left[\frac{-2p}{(\mathrm{p}^2+\mathrm{a}^2)^2}\right] = -t\frac{1}{a}\sin at$

$\therefore L^{-1}\left[\frac{p}{(\mathrm{p}^2+\mathrm{a}^2)^2}\right] = \frac{t}{2a}\sin at$

$\therefore \mathrm{L}^{-1}\left\{\frac{\mathrm{p}^2}{(\mathrm{p}^2+\mathrm{a}^2)^2}\right\} = \frac{d}{dt}\left(\frac{t}{2a}\sin at\right) = \frac{1}{2a}(at\cos at + \sin at)$

Practice Questions:

Find the inverse Laplace transform of

1. $\frac{1}{p(p+2)^3}$ 2. $\frac{1}{p^2(p^2-a^2)}$ 3. $\frac{1}{p^2(p+3)}$ 4. $\frac{1}{p^2(p+1)^2}$

Reference Books:

- Integral transforms by Ian Sneddon, Mc Graw hill publications.
- Introduction to Applied Mathematics by Gilbert Strang, Cambridge Press
- Laplace and Fouries transforms by Dr.J.K. Goyal and K.P. Guptha, Pragathi Prakashan, Meerut.
- The Laplace Transform: Theory and Applications byJoel L.Schiff
- The Laplace Transform by David Vernon Widder
- Student’s Guide to Laplace Transforms (Student's Guides) by Daniel Fleisch.


thank you